Thermodynamic efficiency of self-organization: inferential and information-geometric perspectives

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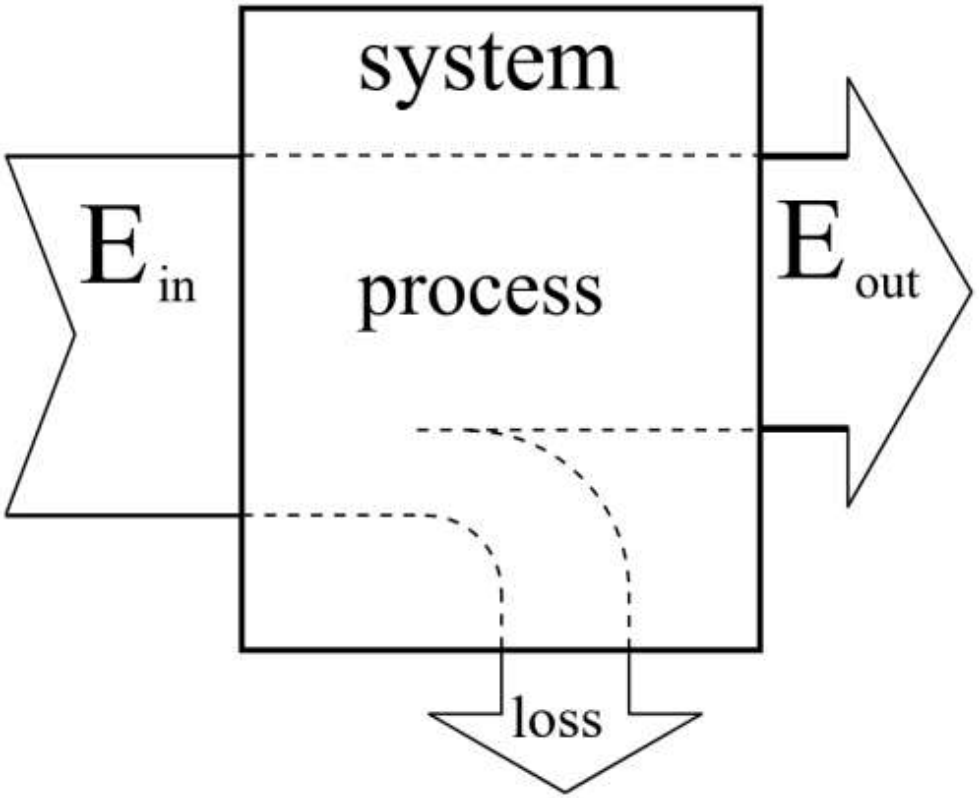
THE UNIVERSITY OF SYDNEY

Santa Fe Institute
15 September 2026

Outline

- Background:
  - Thermal vs thermodynamic efficiency
  - Fisher information and order parameters
- Examples:
  - collective / swarming motion (*Physical Review E*, 2018)
  - Curie-Weiss Ising model (*Entropy*, 2021)
  - 2D Ising model (*Royal Society Open Science*, 2025)
- System-centric vs observer-centric perspectives
- Directional efficiency (*Journal of Physics: Complexity*, 2026)

Thermal efficiency



$$Q_{in} = |W_{\text{out}}| + |Q_{\text{out}}|$$

$$\eta_{\text{th}} \equiv \frac{\text{benefit}}{\text{cost}}$$

$$\eta_{\text{th}} \equiv \frac{|W_{\text{out}}|}{Q_{\text{in}}}$$

https://en.wikipedia.org/wiki/Thermal_efficiency

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Assumptions and clarifications

- the control parameters θ are varied quasi-statically
- the system passes through a sequence of equilibrium states
- the generalised work is the reversible work required to move between neighbouring equilibrium states, and is equal to the corresponding free-energy change

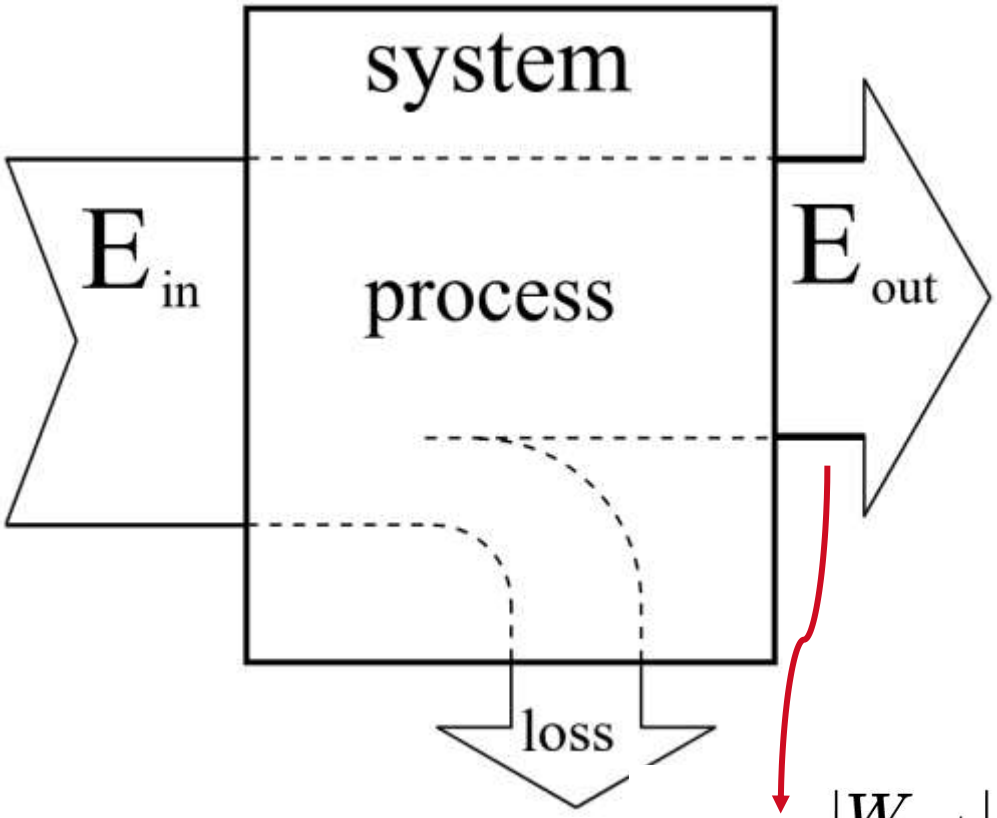
$$W_{\mathrm{gen}}(\theta) = \int_{\theta_{\mathrm{ref}}}^{\theta} dG = G(\theta) - G(\theta_{\mathrm{ref}})$$

- the infinitesimal generalised work

$$dW_{\mathrm{gen}} = dG$$

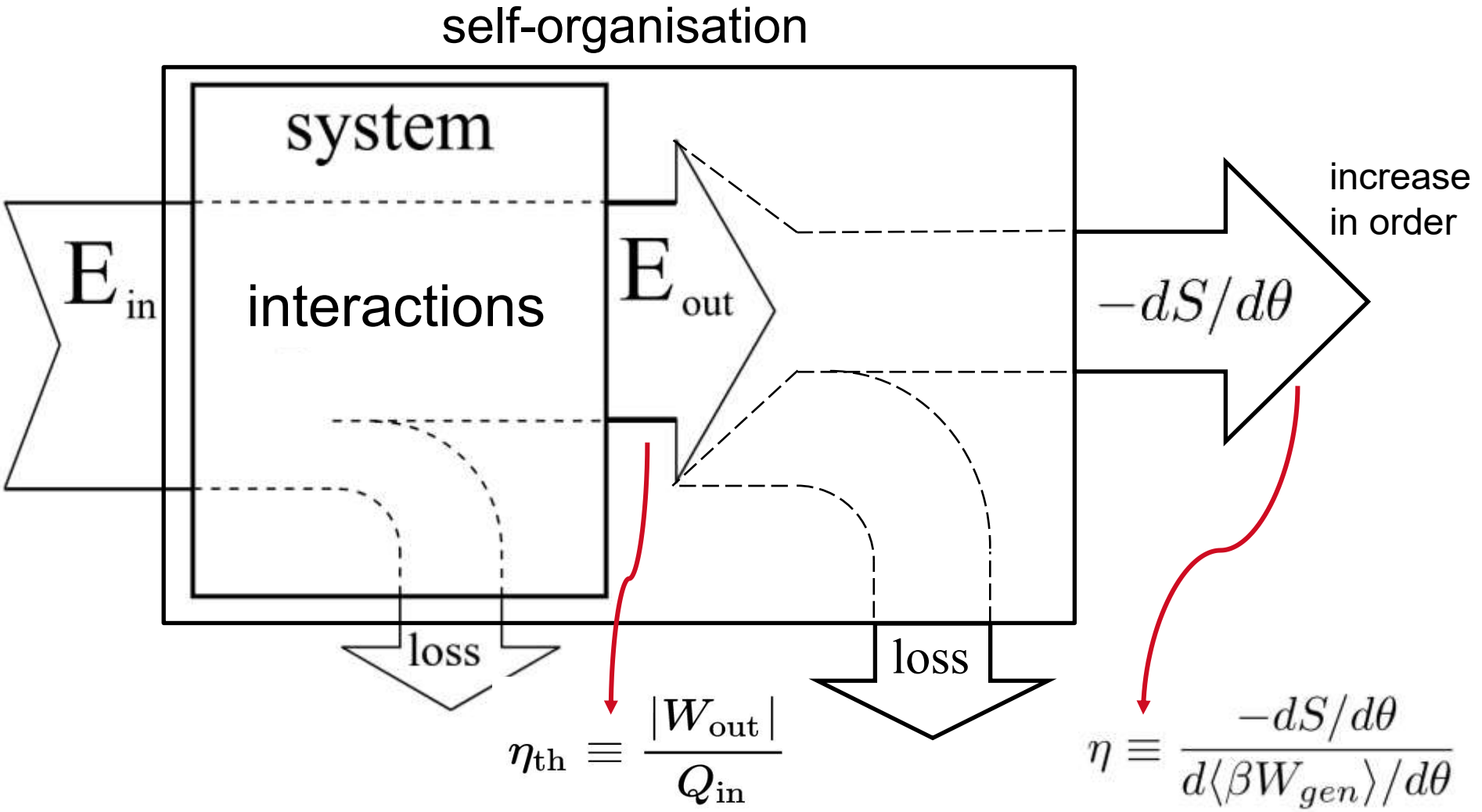
- exponential family probability distributions of the system's states

Thermal efficiency



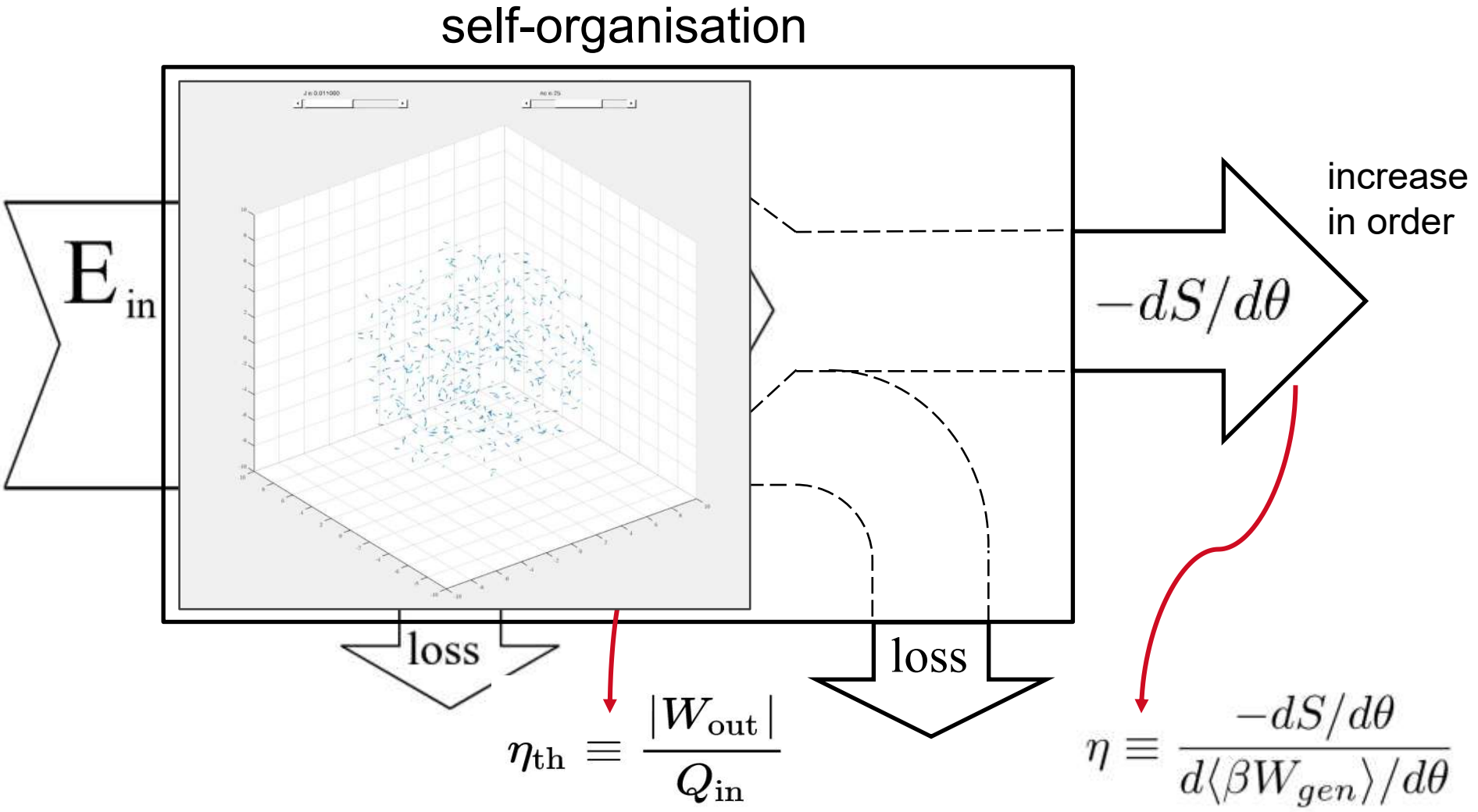
$$\eta_{\text{th}} \equiv \frac{|W_{\text{out}}|}{Q_{\text{in}}}$$

Thermodynamic efficiency



E. Crosato, R. Spinney, R. Nigmatullin, J. T. Lizier, M. Prokopenko, Thermodynamics of collective motion near criticality, *Physical Review E,* 97, 012120, 2018.

Thermodynamic efficiency



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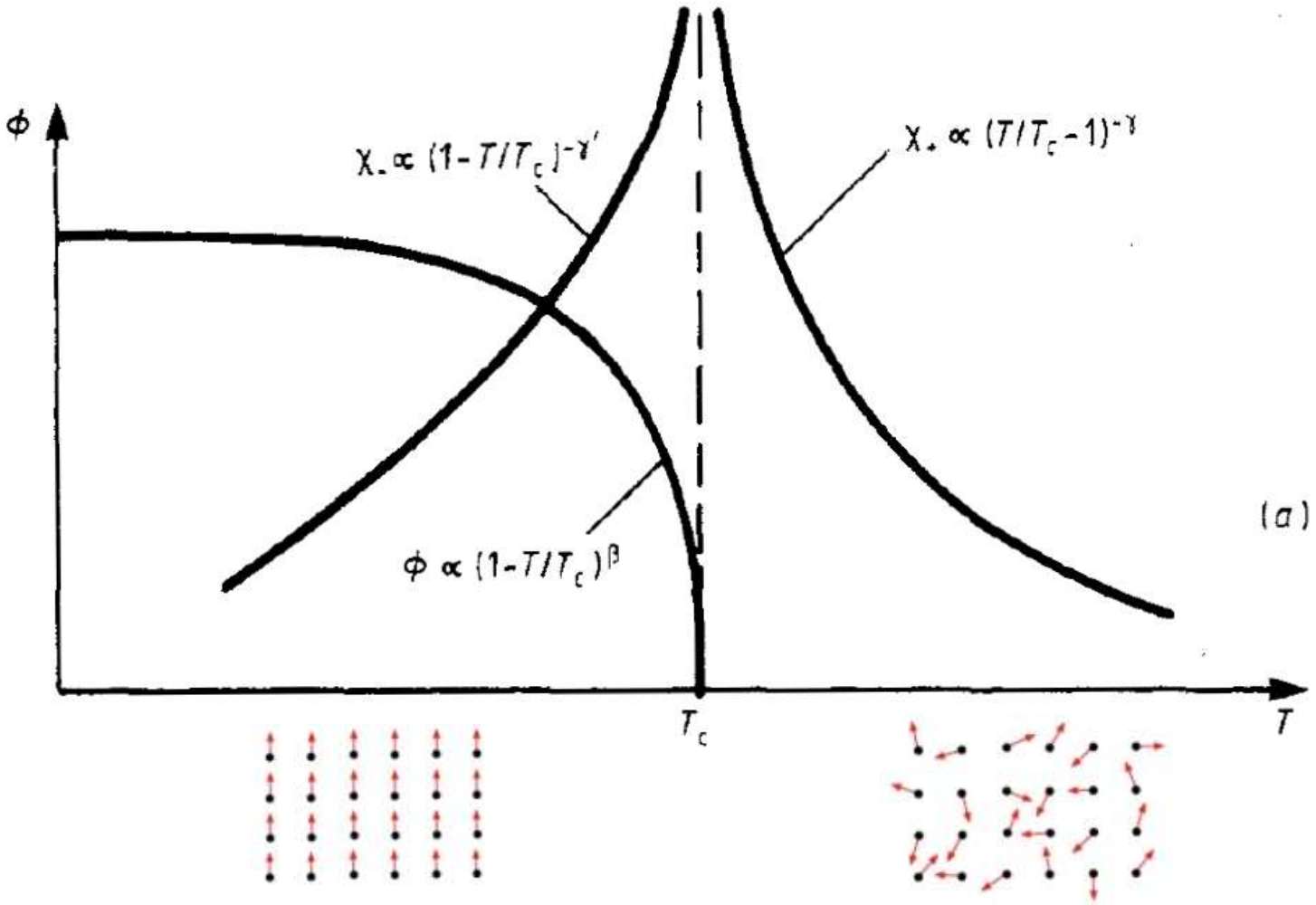
Derivative of order parameter (divergence)

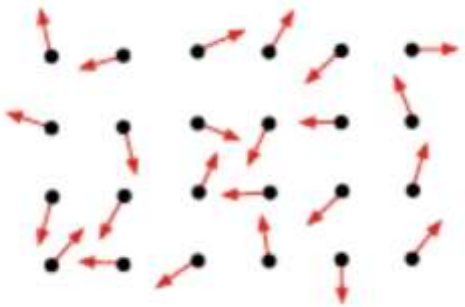
K Binder **(1987)**

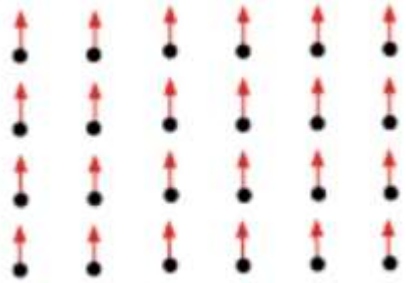


Fisher Information and sensitivity

A way of measuring the amount of information that an observable random variable X has about an unknown parameter θ

$$F(\theta) = \int_x \left(\frac{\partial \ln(p(x|\theta))}{\partial \theta} \right)^2 p(x|\theta) dx$$





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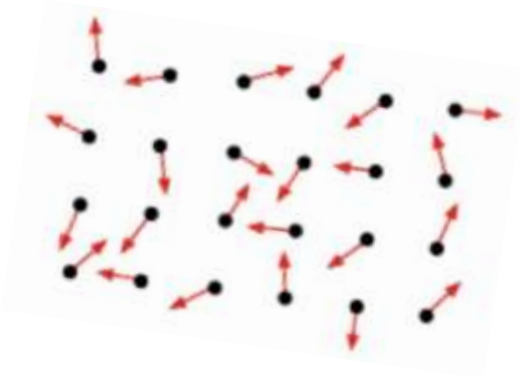

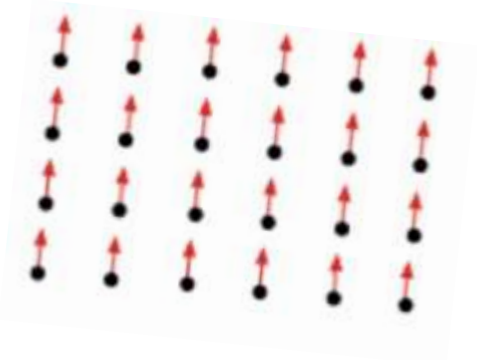

Fisher Information and order parameters

$$G(T,\theta_i) = U(S,\phi_i) - TS - \phi_i\theta_i$$

$$F_{ij}(\theta) = \beta \frac{\partial \phi_i}{\partial \theta_j}$$

Fisher information matrix

Rate of change of the order parameter

M. Prokopenko, J. T. Lizier, O. Obst, X. R. Wang, Relating Fisher information to order parameters, *Physical Review E*, 84, 041116, 2011.

Fisher Information and generalised work

- Fisher information is related not to the absolute amount of work stored in a state, but to how rapidly the required reversible work changes as the control parameter θ varies

- Fisher information is proportional to the curvature of (reversible) work in quasi-static processes

$$F(\theta) = -\frac{d^2 \langle \beta W_{gen} \rangle}{d\theta^2}$$

E. Crosato, R. Spinney, R. Nigmatullin, J. T. Lizier, M. Prokopenko, Thermodynamics of collective motion near criticality, *Physical Review E,* 97, 012120, 2018.

Thermodynamic efficiency

The reduction in uncertainty (the increase in order) from an expenditure of work for a given value of control parameter

$$\eta \equiv \frac{-dS/d\theta}{d\langle \beta W_{gen} \rangle / d\theta}$$

in thermodynamic terms

E. Crosato, R. Spinney, R. Nigmatullin, J. T. Lizier, M. Prokopenko, Thermodynamics of collective motion near criticality, *Physical Review E,* 97, 012120, 2018.

Thermodynamic efficiency

$$F(\theta) = -\frac{d^2\langle \beta W_{gen} \rangle}{d\theta^2}$$

$$\eta \equiv \frac{-dS/d\theta}{d\langle \beta W_{gen} \rangle / d\theta} = \frac{-dS/d\theta}{\int_{\theta}^{\theta^*} F(\theta')d\theta'}$$

in thermodynamic terms

in computational terms

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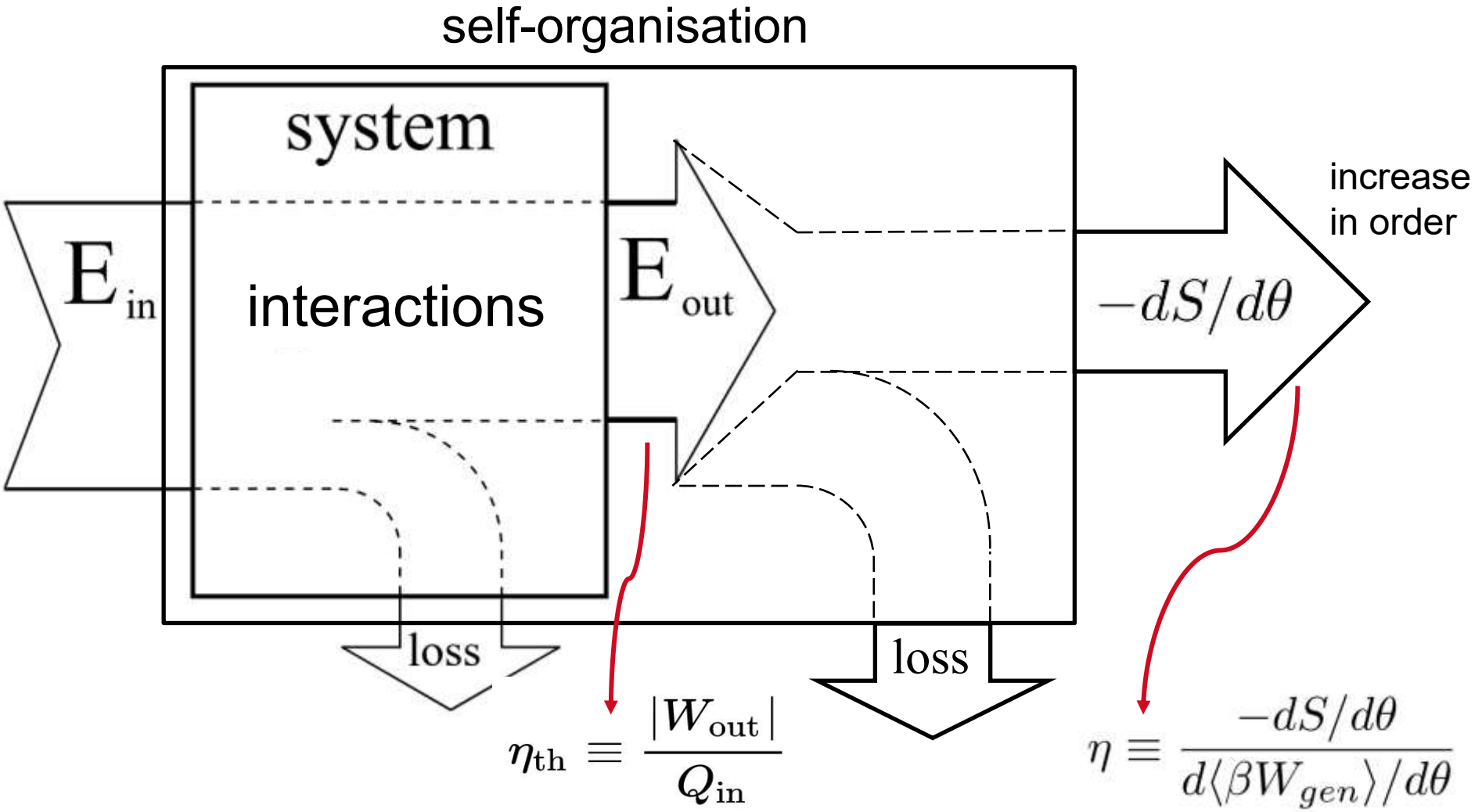
Thermodynamic efficiency



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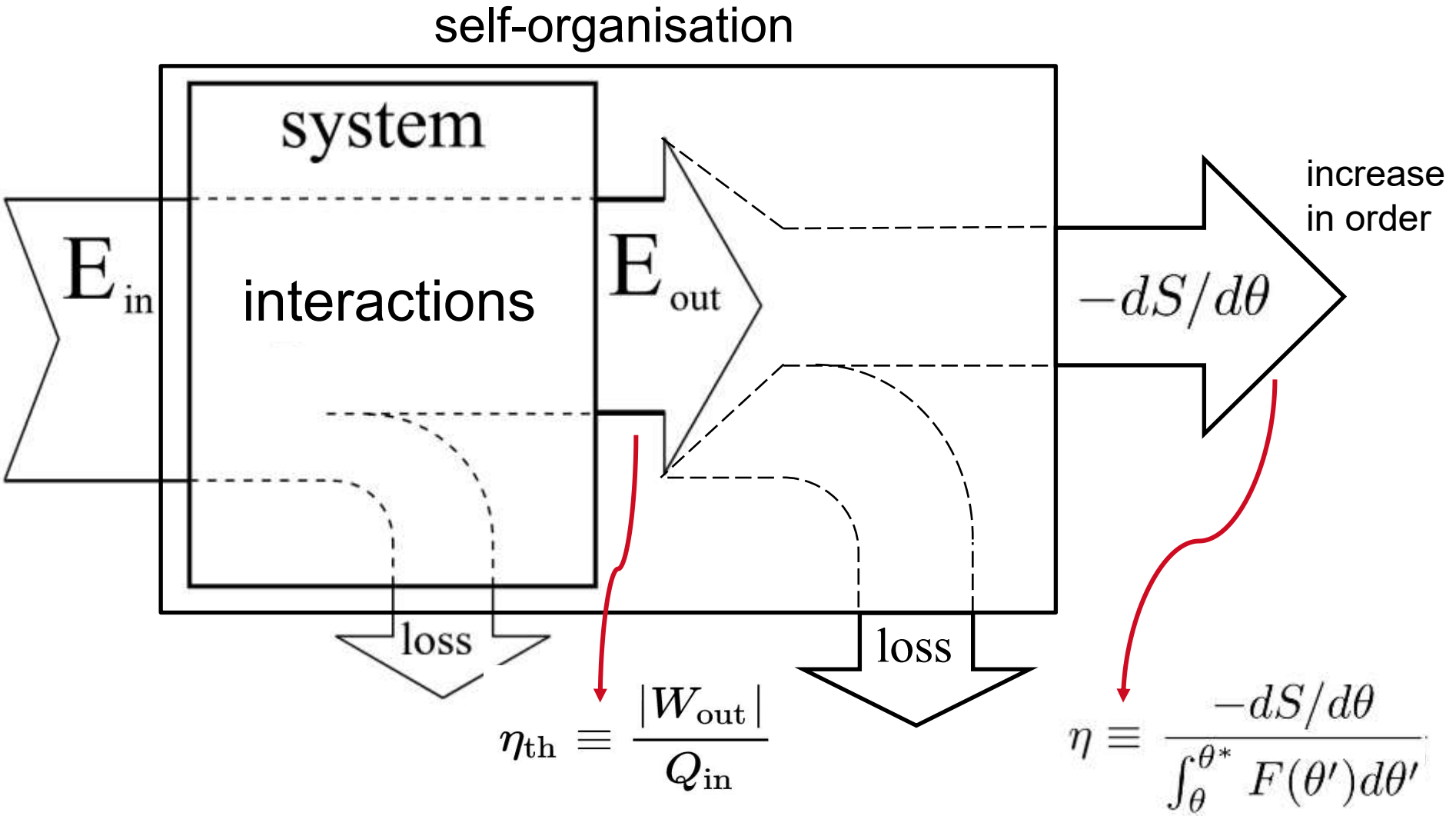
Thermodynamic efficiency



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1. A dynamical model of collective motion

VOLUME 92, NUMBER 2 PHYSICAL REVIEW LETTERS week ending 16 JANUARY 2004

Onset of Collective and Cohesive Motion

Guillaume Grégoire and Hugues Chaté
CEA–Service de Physique de l'État Condensé, CEN Saclay, 91191 Gif-sur-Yvette, France
Pôle Matière et Systèmes Complexes, CNRS FRE 2348, Université de Paris VII, Paris, France
(Received 12 August 2003; published 15 January 2004)

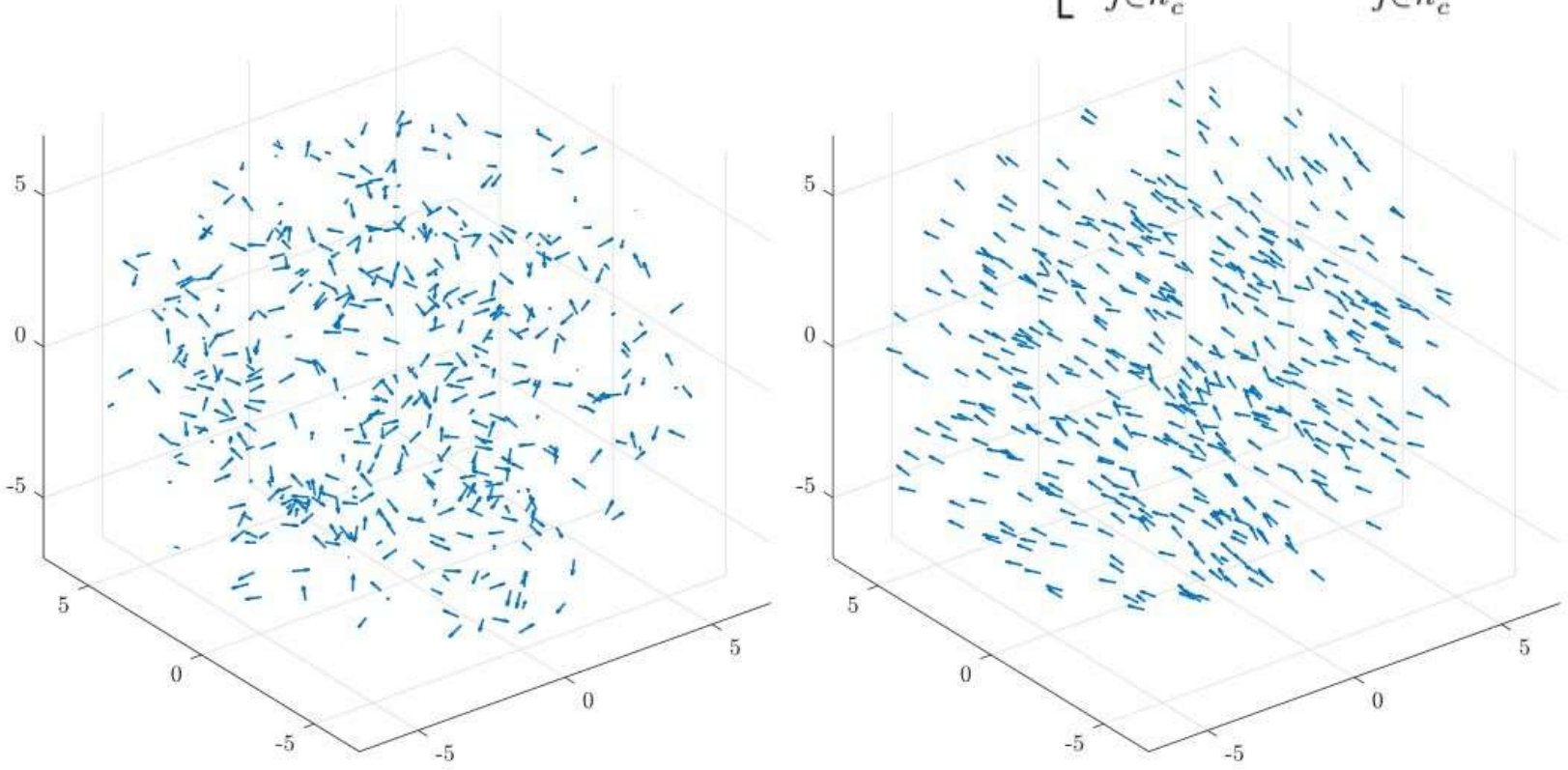
$$\mathbf{x_i}(t+1) = \mathbf{x_i}(t) + \mathbf{v_i}(t)$$

$$\mathbf{v_i}(t+1) = v_0 \Theta \left[a \sum_{j \in n_c^i} \mathbf{v_j}(t) + b \sum_{j \in n_c^i} f_{ij} + n_c \boldsymbol{\eta_i} \right]$$

alignment cohesion perturbation

Swarming (collective) motion

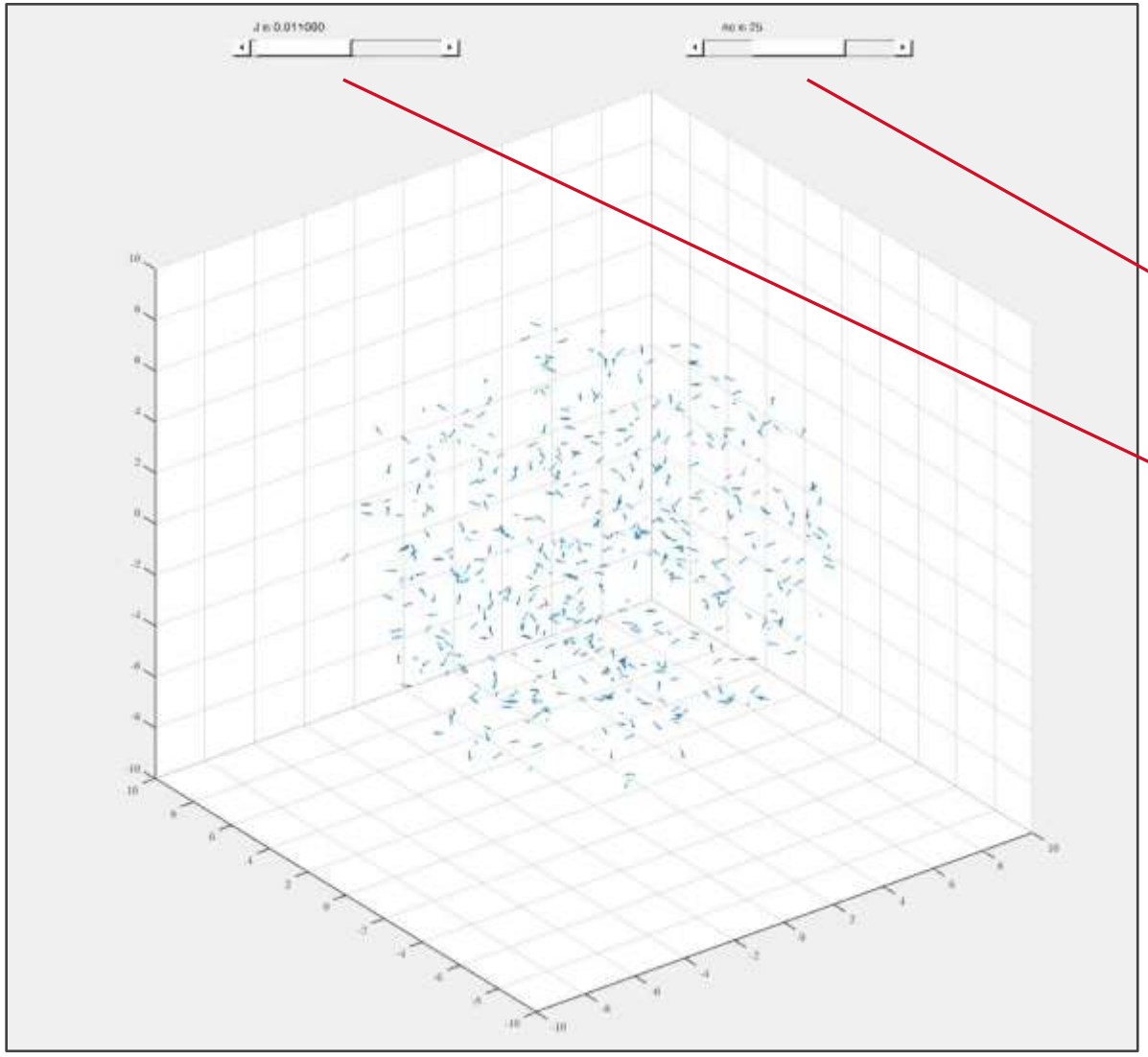
$$\mathbf{v_i}(t+1) = v_0 \Theta \left[a \sum_{j \in n_c^i} \mathbf{v_j}(t) + b \sum_{j \in n_c^i} f_{ij} + n_c \boldsymbol{\eta_i} \right]$$



(a) $J = 0.001$, $n_c = 20$ (b) $J = 0.2$, $n_c = 20$

E. Crosato, R. Spinney, R. Nigmatullin, J. T. Lizier, M. Prokopenko, Thermodynamics of collective motion near criticality, *Physical Review E,* 97, 012120, 2018.

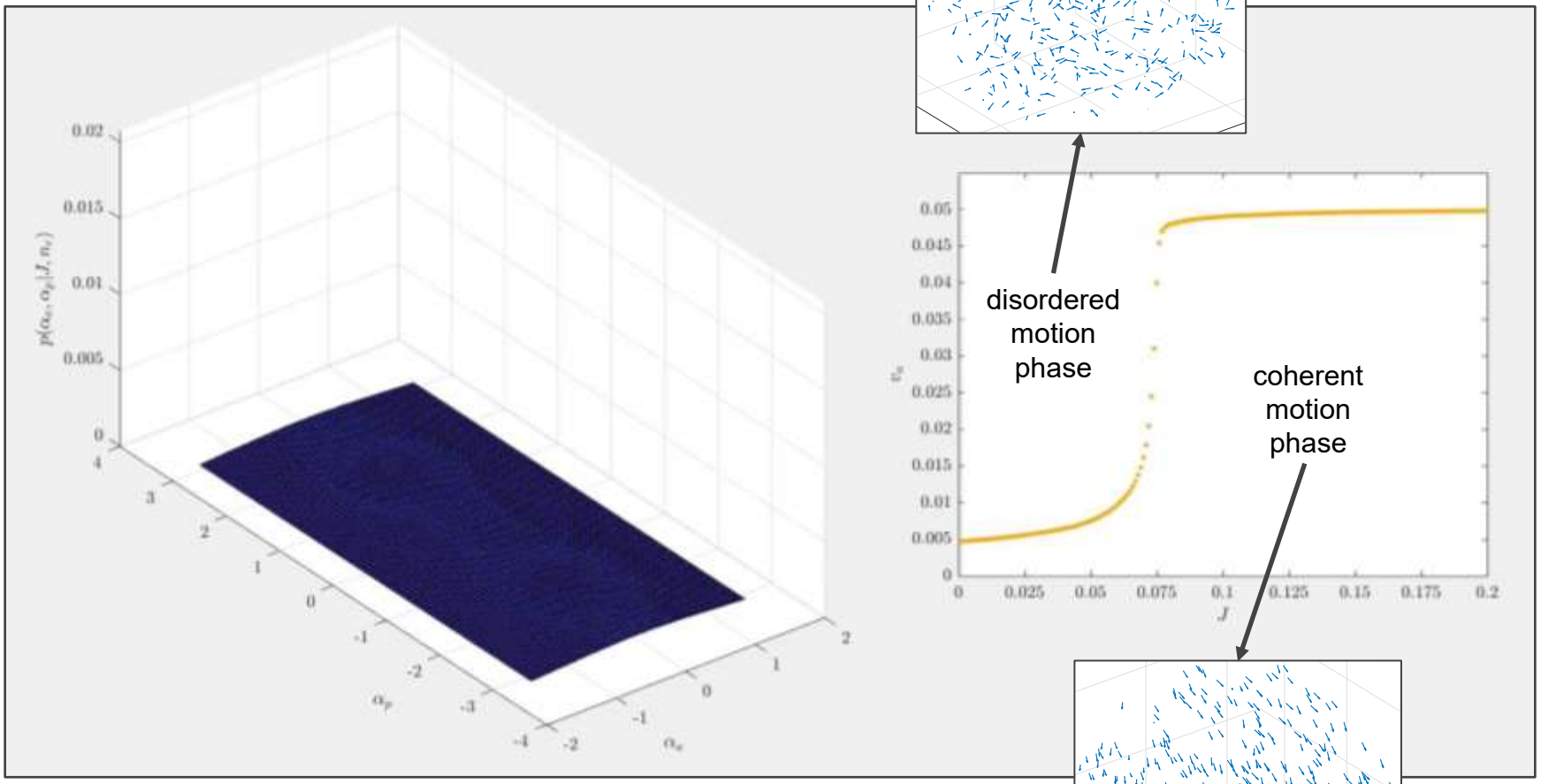
Varying control parameters

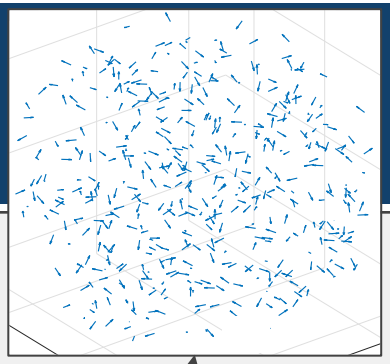


a kinetic phase transition driven by

- nearest neighbours N_c
- alignment strength $J = v_0 a$

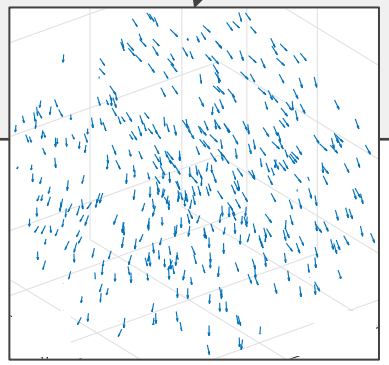
The two kinetic phases





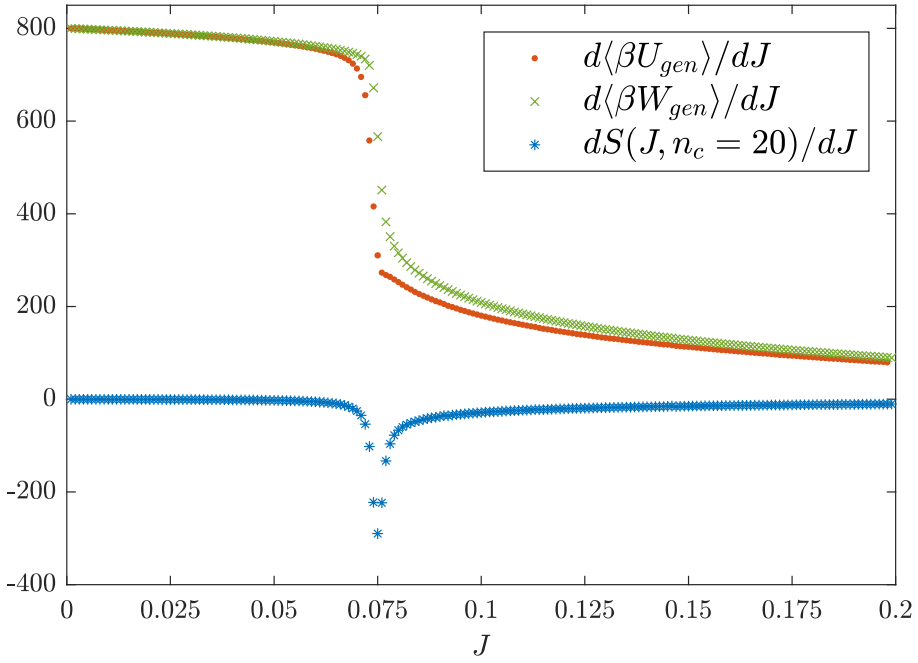
disordered motion phase

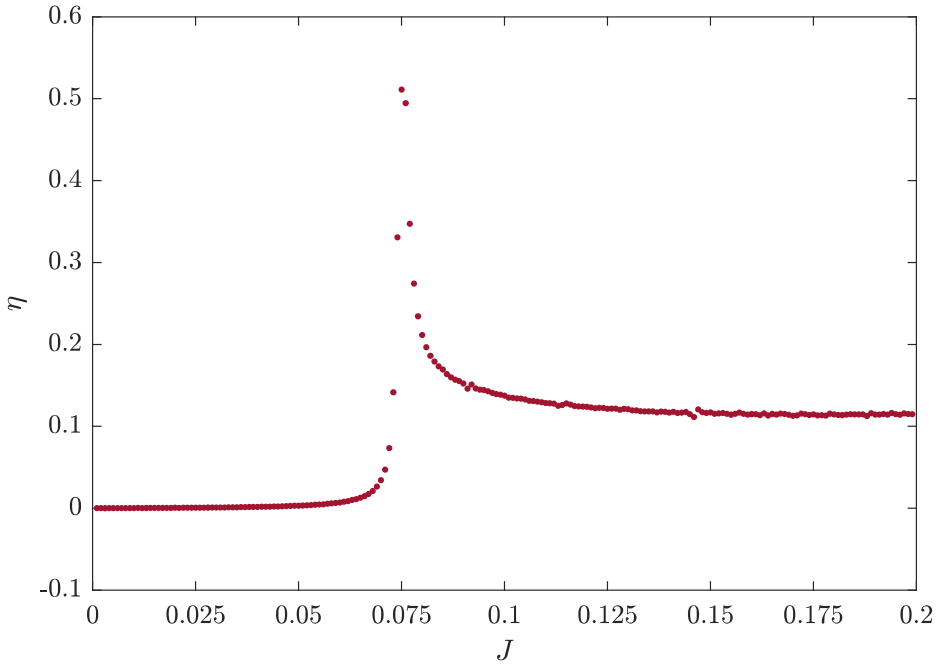
coherent motion phase



Thermodynamic efficiency of swarming behaviour

$$\eta = \frac{-dS(J, n_c)/dJ}{d\langle \beta W_{gen} \rangle / dJ}$$





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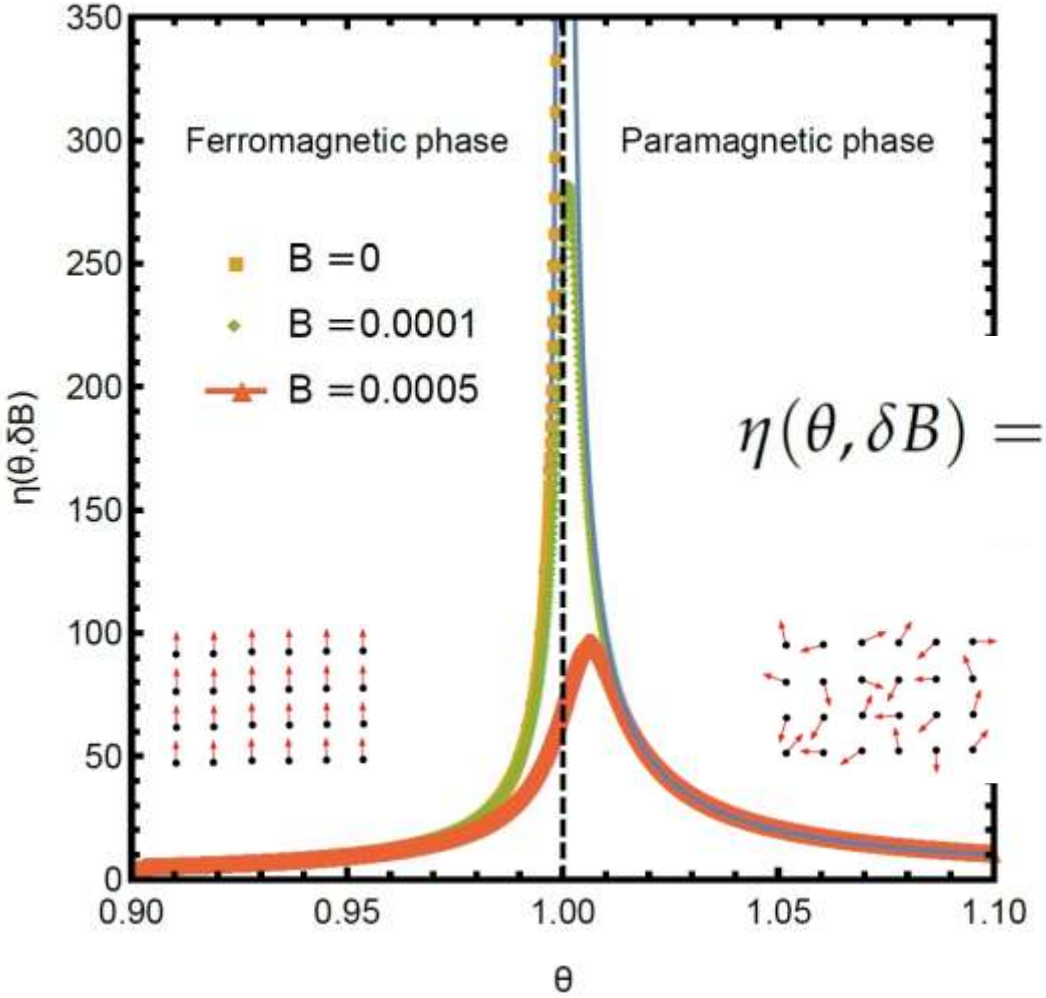
2. Back to magnets… Curie-Weiss Ising model

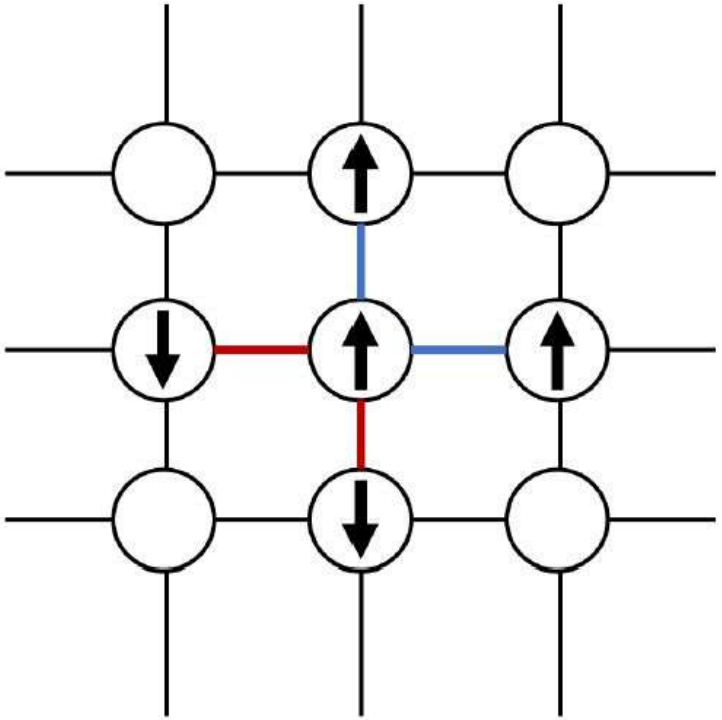


$$t \equiv (\theta - \theta_c)/\theta_c$$

$$\eta(\theta, \delta B) = \begin{cases} -\frac{1}{k_B}\frac{1}{2}t^{-1} & \text{for } t < 0, \\ \frac{1}{k_B \theta_c} t^{-1} & \text{for } t > 0. \end{cases}$$

R. Nigmatullin, M. Prokopenko, Thermodynamic efficiency of interactions in self-organizing systems, *Entropy*, 23(6): 757, 2021.

3. 2D Ising model



$$\sigma^{(i)}\sigma^{(j)} = \begin{cases} +1 & \text{if sites i, j aligned} \\ -1 & \text{if sites i, j misaligned} \end{cases}$$

$$E^{(i)} = \sum_{j \in \nu^{(i)}} -J\sigma^{(i)}\sigma^{(j)}$$

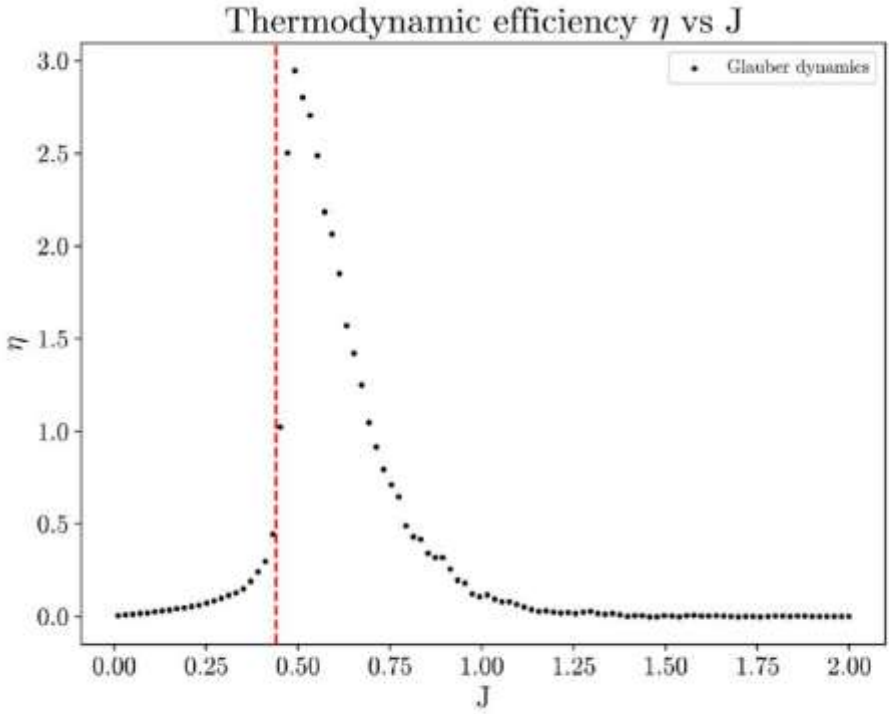
$$dE^{(i)} = E^{(i)}(\text{after flip}) - E^{(i)}(\text{before flip})$$

$$= \sum_{j \in \nu^{(i)}} 2J\sigma^{(i)}\sigma^{(j)}.$$

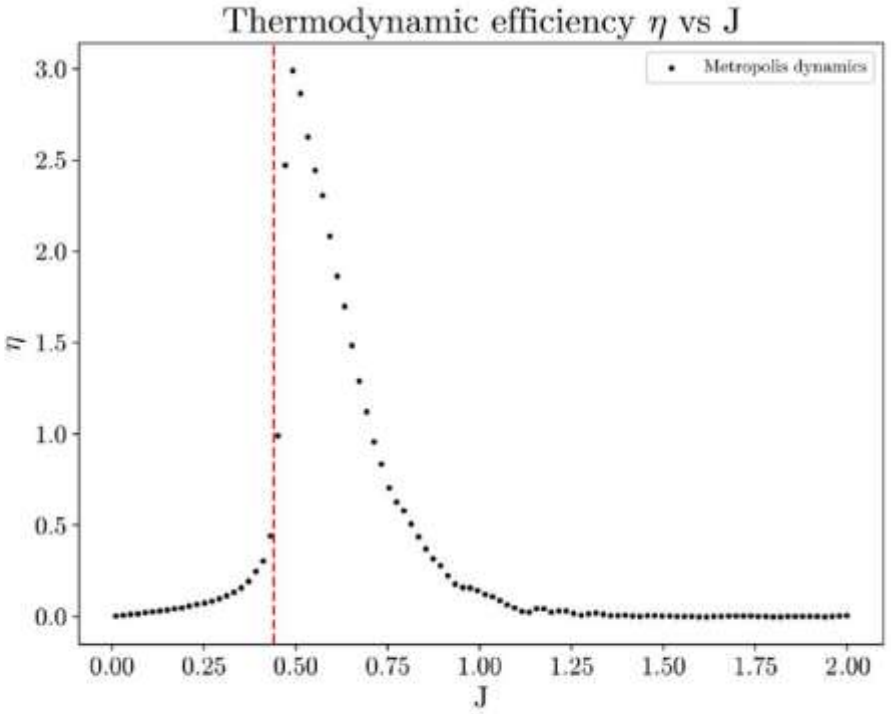
$$p_G(\text{flip}) = 0.5\left[1 - \tanh(0.5\beta dE^{(i)})\right] \qquad p_M(\text{flip}) = \min\left[1, e^{-\beta dE^{(i)}}\right]$$

Q. Chen, M. Prokopenko, Why collective behaviours self-organize to criticality: a primer on information-theoretic and thermodynamic utility measures, *R. Soc. Open Sci.*, 12: 241655, 2025.

3. 2D Ising model



(a) Glauber dynamics



(b) Metropolis dynamics

$$p_G(\text{flip}) = 0.5\left[1 - \tanh(0.5\beta dE^{(i)})\right]$$

$$p_M(\text{flip}) = \min\left[1, e^{-\beta dE^{(i)}}\right]$$

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Principle of super-efficiency

- *Self-organizing collective systems strive to maximize the thermodynamic efficiency of interactions by gaining maximal predictability of collective behaviour per unit of the expended work. This efficiency is maximized at the critical regime:*

$$\theta^* = \arg\max_{\theta} \{\eta(\theta)\}$$

$$= \arg\max_{\theta} \frac{-\partial \mathbb{S}/\partial\theta}{\partial\langle\beta \mathbb{W}_{gen}\rangle/\partial\theta}$$

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Summary (1/3): System-centric view

- Thermodynamic and computational perspectives:
  - ratio of the reduction in uncertainty to the rate of work required to change control parameter (= accumulated sensitivity of distributed computation)

$$\eta^{\dagger}\left(\lambda_j\right) = -\frac{\partial S\left(\underline{\lambda}\right)}{\partial \lambda_j} \Big/ \int_{\lambda_j^*}^{\lambda_j} \mathcal{I}_{jj}\left(\lambda_1, \ldots, \lambda_j', \ldots, \lambda_n\right) \mathrm{d}\lambda_j'$$

  - diverges at critical point for model systems (e.g., Curie-Weiss model or 2D Ising model)

$$\eta(X, \delta X) = -\frac{1}{k_B} \frac{\beta}{|T - T_c|} \qquad \eta(J) = \frac{\ln(1+\sqrt{2})}{2} |J - J_c|^{-1}$$

Observer-centric view: inferential form

- Thermodynamic efficiency can be expressed via fluctuations in the observables *X* driven by changing a specific control parameter λ (inferential efficiency):

multiple parameters:

$$\eta^{\bowtie}(\lambda_j) = -\frac{\sum_i \lambda_i \mathrm{Cov}(X_i, X_j)}{\langle X_j \rangle}$$

single parameter:

$$\eta^{\bowtie}(\lambda) = -\frac{\lambda \mathrm{Var}(X)}{\langle X \rangle}$$

Q. Chen, N. Ay, M. Prokopenko, Generalising thermodynamic efficiency of interactions: inferential, information-geometric and computational perspectives, *J. Phys. Complexity*, 7: 025002, 2026.

Observer-centric view: information-geometric form

- Potentially higher precision (smaller lower bound of the estimation variance) is contrasted with potentially higher total precision accumulated along the protocol:

multiple parameters:

$$\eta^{\triangle}\left(\lambda_j\right) = \frac{\sum_i \lambda_i \mathcal{I}_{ij}\left(\underline{\lambda}\right)}{\int_{\lambda_j^*}^{\lambda_j} \mathcal{I}_{jj}\left(\lambda_1, \ldots, \lambda_j', \ldots, \lambda_n\right) \mathrm{d}\lambda_j'}$$

single parameter:

$$\eta^{\triangle}\left(\lambda\right) = \frac{\lambda \mathcal{I}\left(\lambda\right)}{\int_{\lambda^*}^{\lambda} \mathcal{I}\left(\lambda'\right) \mathrm{d}\lambda'}$$

$$\mathrm{Cov}\left(\underline{\lambda}\right) \geqslant \mathcal{I}^{-1}\left(\underline{\lambda}\right)$$

Q. Chen, N. Ay, M. Prokopenko, Generalising thermodynamic efficiency of interactions: inferential, information-geometric and computational perspectives, *J. Phys. Complexity*, 7: 025002, 2026.

Summary (2/3): Observer-centric view

- Inferential form connects the efficiency of self-organisation to the efficiency of statistical inference:
  - computes normalized weighted sum of variance and covariances of the macroscopic observables
  - captures how changing a specific control parameter λ_j influences the joint fluctuations of all the observables $\{X_i\}$
- Information-geometric form contrasts potentially higher precision (smaller lower bound of the estimation variance) with potentially higher total precision accumulated along the protocol
- Both forms capture the observer's ability to extract information about the system

System-centric vs observer-centric views

$$\eta^{\dagger}\left(\lambda_j\right) = -\frac{\partial S\left(\underline{\lambda}\right)}{\partial \lambda_j} \Big/ \int_{\lambda_j^*}^{\lambda_j} \mathcal{I}_{jj}\left(\lambda_1, \ldots, \lambda_j', \ldots, \lambda_n\right) \mathrm{d}\lambda_j'$$

$$\eta^{\bowtie}\left(\lambda_j\right) = -\frac{\sum_i \lambda_i \operatorname{Cov}\left(X_i, X_j\right)}{\langle X_j \rangle}$$

$$\eta^{\triangle}\left(\lambda_j\right) = \frac{\sum_i \lambda_i \mathcal{I}_{ij}\left(\underline{\lambda}\right)}{\int_{\lambda_j^*}^{\lambda_j} \mathcal{I}_{jj}\left(\lambda_1, \ldots, \lambda_j', \ldots, \lambda_n\right) \mathrm{d}\lambda_j'}$$

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Directional efficiency

- Generalised directional efficiency evaluated at $\lambda = (\lambda_1, \ldots, \lambda_n)$ for the protocol moving in the direction $v = (v_1, \ldots, v_n)$ is defined as:

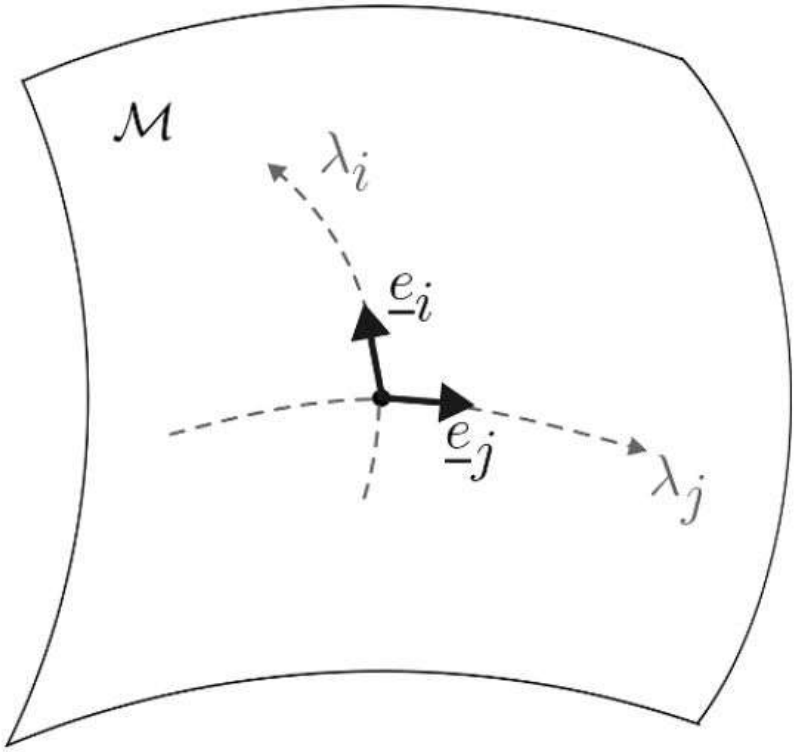
$$\eta_{\underline{v}}(\underline{\lambda}) = \frac{-D_{\underline{v}} S(\underline{\lambda})}{D_{\underline{v}} \langle \beta \mathbb{W} \rangle (\underline{\lambda})}$$

$$D_{\underline{v}} f(\underline{\lambda}) := \lim_{t \to 0} \frac{f(\underline{\lambda} + t\underline{v}) - f(\underline{\lambda})}{t} = \nabla f \cdot \underline{v}$$

Q. Chen, N. Ay, M. Prokopenko, Generalising thermodynamic efficiency of interactions: inferential, information-geometric and computational perspectives, *J. Phys. Complexity*, 7: 025002, 2026.

Directional efficiency

- λ_i, λ_j are coordinate curves, and e_i, e_j are the corresponding tangent vectors. A point on the manifold is a probability density $p(x;\lambda)$ of a random variable x, parameterized by λ:



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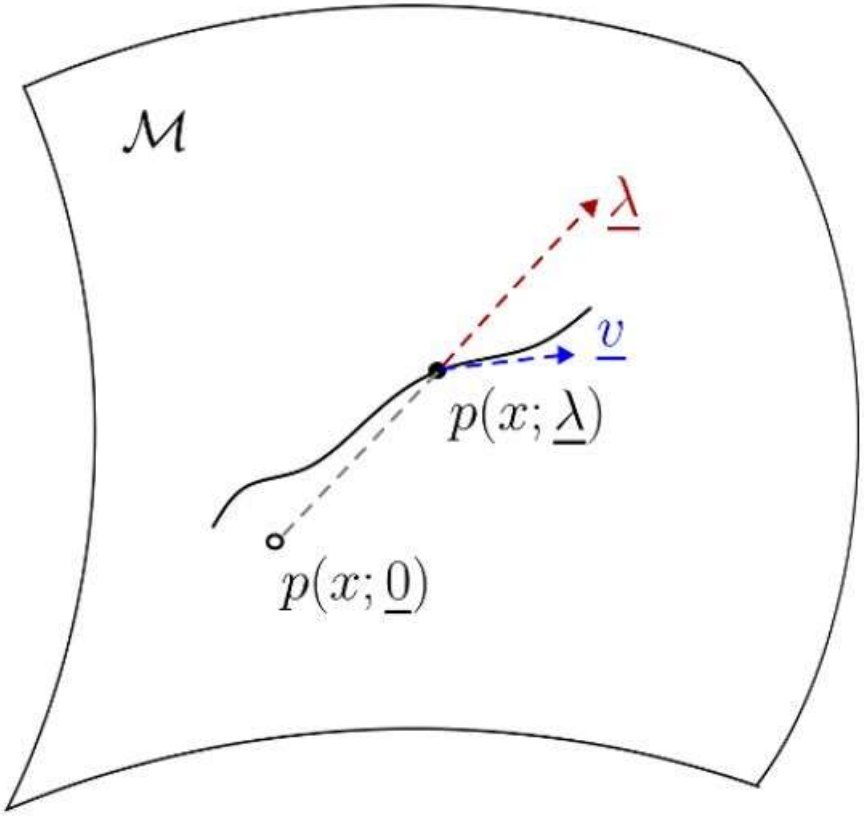
Directional efficiency

- The steepest direction on the statistical manifold is given by the natural gradient of −S(λ):



$$-\mathcal{I}^{-1}(\underline{\lambda})\nabla S(\underline{\lambda}) = \underline{\lambda}$$

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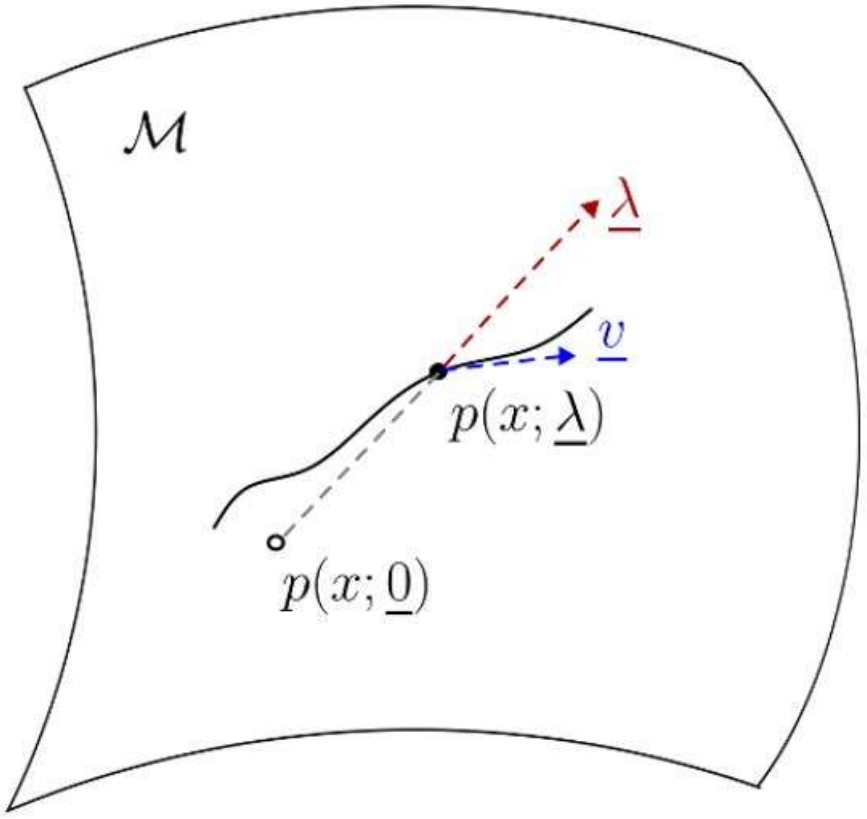


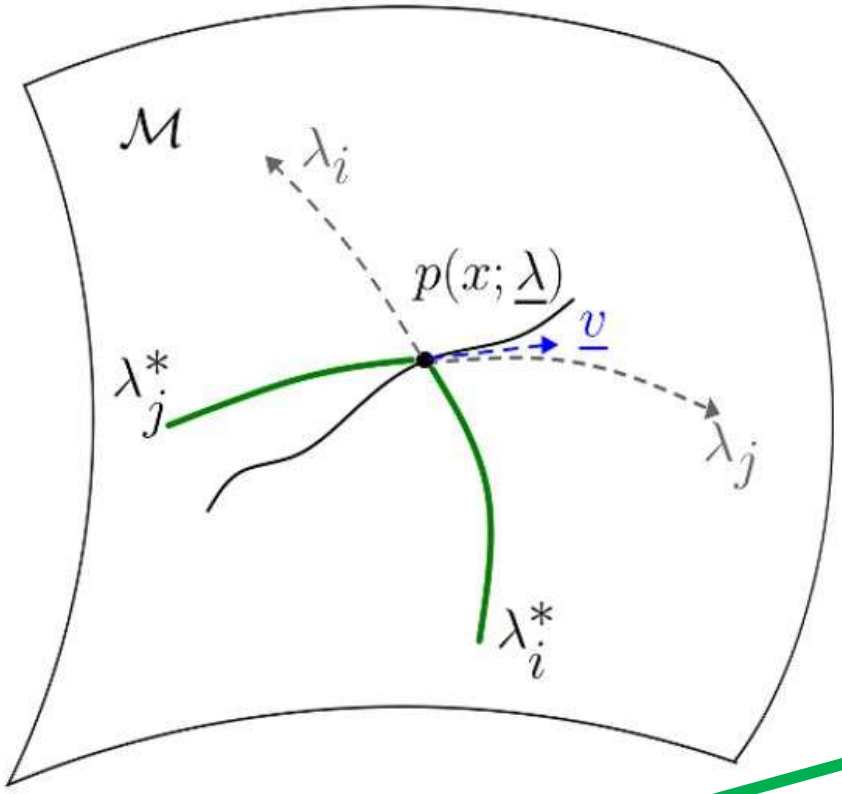
$$\eta_{\underline{\nu}}(\underline{\lambda}) = \frac{-D_{\underline{\nu}}S(\underline{\lambda})}{D_{\underline{\nu}}\langle \beta \mathbb{W} \rangle (\underline{\lambda})}$$

$$-D_{\underline{\nu}}S(\underline{\lambda}) = \langle \underline{\nu}, \underline{\lambda} \rangle$$

the extent to which the protocol's velocity aligns with the most efficient direction

Q. Chen, N. Ay, M. Prokopenko, Generalising thermodynamic efficiency of interactions: inferential, information-geometric and computational perspectives, *J. Phys. Complexity*, 7: 025002, 2026.

Directional efficiency



$$E(\gamma) := \frac{1}{2}\int_0^1 \dot{\gamma}(t)^\top \mathcal{I}_{\gamma(t)}\dot{\gamma}(t)\,\mathrm{d}t$$

$$\eta_{\underline{v}}(\underline{\lambda}) = \frac{-D_{\underline{v}}S(\underline{\lambda})}{D_{\underline{v}}\langle\beta\mathbb{W}\rangle(\underline{\lambda})}$$

$$D_{\underline{v}}\langle\beta\mathbb{W}\rangle(\underline{\lambda}) = \sum_j v_j \int_{\lambda_j^*}^{\lambda_j} \mathcal{I}_{jj}\left(\lambda_1,\ldots,\lambda_j',\ldots,\lambda_n\right)\mathrm{d}\lambda_j'$$

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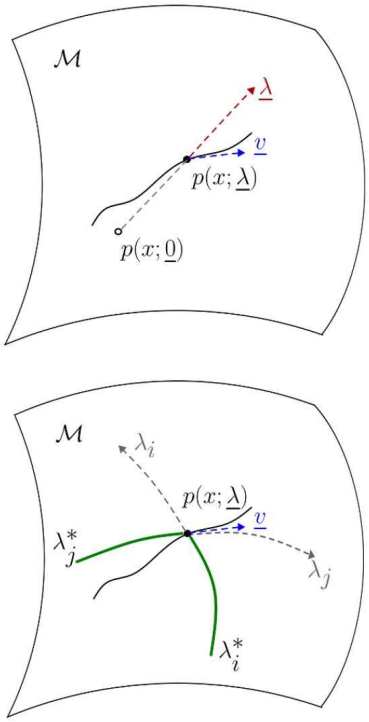
Directional efficiency

Directional efficiency is the ratio of

- (a) the extent to which the protocol's velocity aligns with the most efficient direction to
- (b) the weighted sum of the energies of the paths from the zero-response point to the current point over all dimensions:

$$\eta_{\underline{v}}(\underline{\lambda}) = \frac{\langle \underline{v}, \underline{\lambda} \rangle}{\sum_j \nu_j \int_{\lambda_j^*}^{\lambda_j} \mathcal{I}_{jj}\left(\lambda_1, \ldots, \lambda_j', \ldots, \lambda_n\right) \mathrm{d}\lambda_j'}$$



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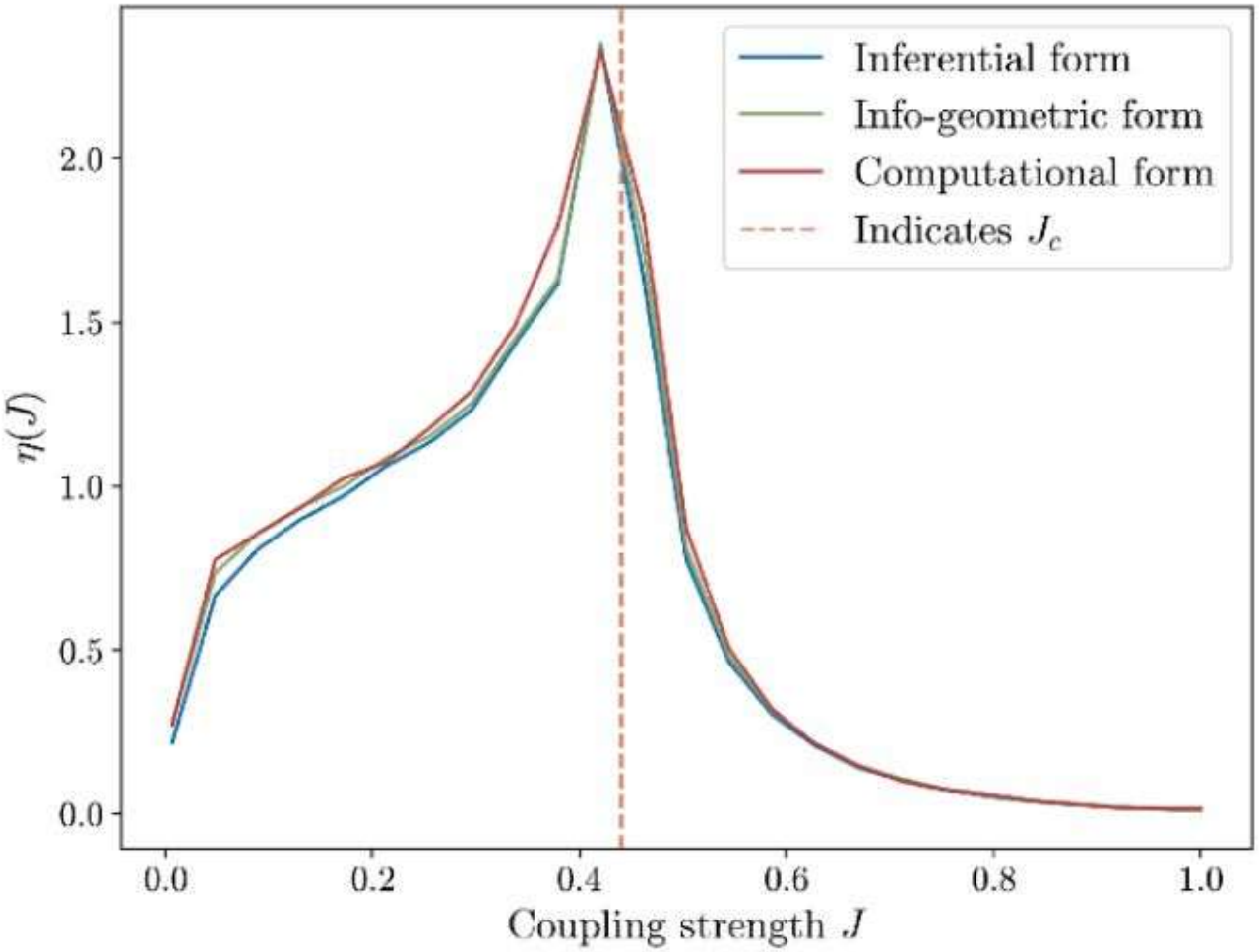
2D Ising model



Q. Chen, N. Ay, M. Prokopenko, Generalising thermodynamic efficiency of interactions: inferential, information-geometric and computational perspectives, *J. Phys. Complexity*, 7: 025002, 2026.

Summary (3/3): Directional efficiency

- Interpreted through the lens of navigation on the statistical manifold following a control protocol trajectory
- Given a protocol, it evaluates how closely the path direction aligns with the natural gradient of entropy descent, relative to the weighted sum of path energies across all coordinate axes of the manifold
- A protocol with high efficiency balances the rate of entropy reduction with the associated total energy
- Connects to parameter space compression, where moving in a stiffer direction leads to a stronger effect on the system's response, corresponding to higher efficiency

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A teaser…

PHYSICAL REVIEW RESEARCH **8**, 033265 (2026)

Thermodynamic efficiency of self-organization in nonequilibrium steady states

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Active matter generates order or patterns through nonequilibrium dynamics. An open research challenge is to determine how efficiently a nonequilibrium self-organizing system can convert consumed energy into macroscopic order. We study an information-theoretic quantity that directly addresses this challenge by estimating the entropy reduction induced by a small control-parameter perturbation, relative to the generalized work required for the perturbation. This quantity has previously been considered mainly in an equilibrium or near-equilibrium context, and here we extend this framework and apply it to two nonequilibrium self-organizing systems: persistent and active Ising models. We observe that the thermodynamic efficiency of nonequilibrium systems maximizes at phase transitions, as in equilibrium systems. Furthermore, we compare thermodynamic efficiency and inferential efficiency across control parameters. While these two quantities are equal in equilibrium as a consequence of the fluctuation-dissipation theorem, we report that they diverge out of equilibrium, and the gap serves as a phenomenological signature of broken detailed balance.

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Conclusions

- Thermodynamic efficiency provides an intrinsic utility to the collective system that guides the system's behaviour independently of the observer
  - ✓ A super-efficient self-organising system approaches the point where it can gain maximal predictability of its collective behaviour, given the amount of additional work available to change the control parameter
  - ✓ Alternatively, given a desired predictability gain, a super-efficient system seeks the point where the energy cost of changing the control parameter is minimal

- The principle of super-efficiency explains *why* it is beneficial for a group of interacting agents to operate at the critical regime, rather than *how* the system could self-organize to this point — the latter question is pursued by SOC models

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