On thermodynamic efficiency of swarming behaviour

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Self-organisation of coherent motion in systems of self-propelled particles (e.g., flocks, swarms, active matter) is a pervasive phenomenon [1, 2, 3, 4, 5, 6, 7], which may be explained by some underlying universal principles. We interpret self-organisation of collective motion as an example of collective and distributed computation, and study it as a thermodynamic phenomenon [8]. In this abstract we report on our investigation of key thermodynamic quantities such as free entropy and generalised work, using the dynamical model of collective motion proposed by Grégoire and Chaté [9]. This model exhibits a kinetic phase transition over the parameters controlling the particles' alignment, separating (i) the "disordered motion" phase, in which particles do not settle on a dominant direction while sharing a collective space, and (ii) the "coherent motion" phase, in which particles cohesively move in a common direction.

The Fisher information [10] measures the amount of information that an observable random variable $\mathcal{X}$ carries about unknown parameters $\theta = [\theta_1, \theta_2, \ldots, \theta_M]$. The probability of the states of the system, described by the state functions $X_m(x)$ over the configuration space and thermodynamic variables θ_m, in a stationary state, is given by the Gibbs measure:

$$p(x|\theta) = \frac{1}{Z(\theta)} e^{-\beta H(x,\theta)} = \frac{1}{Z(\theta)} e^{-\sum_m \theta_m X_m(x)}, \tag{1}$$

where $\beta = 1/k_b T$ is the inverse temperature T (k_b is the Boltzmann constant), the Hamiltonian $H(x, \theta)$ defines the total energy at state x, and $Z(\theta)$ is the partition function [11, 12]. The Gibbs free energy of such system is:

$$G(T, \theta_m) = U(S, \phi_m) - TS - \phi_m \theta_m, \tag{2}$$

where U is the internal energy of the system, S is the configuration entropy and ϕ_m is an order parameter. For a physical system described by the Gibbs measure in Eq. (1), the Fisher information has several physical interpretations, e.g., it is equivalent to the thermodynamic metric tensor $g_{mn}(\theta)$, measures the size of the fluctuations about equilibrium in the collective variables X_m and X_n, is proportional to the curvature of the free entropy $\psi = \ln Z = -\beta G$, and to the derivatives of the corresponding order parameters with respect to the collective variables [11, 13, 14, 12, 15, 16]:

$$F_{mn}(\theta) = g_{mn}(\theta) = \Big\langle (X_m(x) - \langle X_m \rangle)(X_n(x) - \langle X_n \rangle) \Big\rangle = \frac{\partial^2 \psi}{\partial \theta_m \partial \theta_n} = \beta \frac{\partial \phi_m}{\partial \theta_n}, \tag{3}$$

where the angle brackets represent average values over the ensemble.

It has also been argued that the difference between curvatures of the configuration entropy and the free entropy is related to a computational balance between uncertainty and sensitivity [17]. We established a thermodynamic basis for this relationship as follows [8]:

$$\frac{d^2 \langle \beta U_{gen} \rangle}{d\theta^2} = \frac{d^2 S}{d\theta^2} - \frac{\partial^2 \psi}{\partial \theta_m \partial \theta_n} = \frac{d^2 S}{d\theta^2} - F(\theta), \tag{4}$$

where $\langle U_{gen} \rangle = U(S, \phi) - \phi\theta$. This expression can be interpreted as the difference between the curvature of the free entropy, captured by the Fisher information (the sensitivity of the system), and the curvature of the configuration entropy (the uncertainty of the system). Under a quasi-static protocol, the first law of thermodynamics yields another important result for the generalised work W_{gen}:

$$F(\theta) = -\frac{d^2 \langle \beta W_{gen} \rangle}{d\theta^2}. \tag{5}$$

Our results identify critical regimes and show that during the phase transition, where the configuration entropy of the system decreases, the rates of change of the work and of the internal energy also decrease, while their curvatures diverge.

We also consider a measure of the *thermodynamic efficiency of swarming behaviour* (treated as an example of distributed computation), defined, for a given value of the control parameter θ, as the reduction in uncertainty (that is, the increase in the internal order) that resulted from an expenditure of work:

$$\eta \equiv \frac{-dS/d\theta}{d\langle \beta W_{gen} \rangle / d\theta} = \frac{-dS/d\theta}{\int_{\theta}^{\theta^*} F(\theta') d\theta'}, \tag{6}$$

where θ^* is the zero-response point for which small changes incur no work [8]. This ratio can be considered entirely in computational terms as the ratio of increasing order, obtained at θ, to the cumulative sensitivity incurred over a process from the current state θ to the state of perfect order, identified by the zero-response point θ^*.

The *sensitivity* and the *uncertainty* are balanced in each phase (disordered motion or coherent motion). However, at criticality, i.e., during a kinetic phase transition, this balance is broken, and the ratio η, specified by Eq. (6), diverges or peaks in finite-size systems. This indicates that the maximal thermodynamical efficiency of swarming behaviour within the system of self-propelled particles is highest during the phase transition.

References

[1] S. Camazine. *Self-organization in Biological Systems.* Princeton University Press, 2001.

[2] I.D. Couzin. Collective minds. *Nature*, 445(7129):715–715, 02 2007.

[3] T. Vicsek and A. Zafeiris. Collective motion. *Physics Reports*, 517(3–4):71 – 140, 2012.

[4] W. Bialek, A. Cavagna, I. Giardina, T. Mora, E. Silvestri, M. Viale, and A.M. Walczak. Statistical mechanics for natural flocks of birds. *Proceedings of the National Academy of Sciences*, 109(13):4786–4791, 2012.

[5] M. Castellana, W. Bialek, A. Cavagna, and I. Giardina. Entropic effects in a nonequilibrium system: Flocks of birds. *Physical Review E*, 93:052416, 2016.

[6] E. Crosato, L. Jiang, V. Lecheval, J.T. Lizier, X.R. Wang, P. Tichit, G. Theraulaz, and M. Prokopenko. Informative and misinformative interactions in a school of fish. *Swarm Intelligence*, 12(4):283–305, 2018.

[7] A. Ward, T. Schaerf, A.L.J. Burns, J.T. Lizier, E. Crosato, M. Prokopenko, and M. Webster. Cohesion, order and information flow in the collective motion of mixed-species shoals. *Royal Society Open Science*, 5:181132, 12 2018.

[8] E. Crosato, R.E. Spinney, R. Nigmatullin, J.T. Lizier, and M. Prokopenko. Thermodynamics and computation during collective motion near criticality. *Physical Review E*, 97, 2018.

[9] G. Grégoire and H. Chaté. Onset of collective and cohesive motion. *Physical Review Letters*, 92:025702, 2004.

[10] R.A. Fisher. On the mathematical foundations of theoretical statistics. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 222(594-604):309–368, 1922.

[11] D.C. Brody and N. Rivier. Geometrical aspects of statistical mechanics. *Physical Review E*, 51:1006–1011, 1995.

[12] G.E. Crooks. Measuring thermodynamic length. *Physical Review Letters*, 99:100602, 2007.

[13] D.C. Brody and A. Ritz. Information geometry of finite Ising models. *Journal of Geometry and Physics*, 47(2):207–220, 2003.

[14] W. Janke, D.A. Johnston, and R. Kenna. Information geometry and phase transitions. *Physica A: Statistical Mechanics and its Applications*, 336(1–2):181–186, 5 2004.

[15] X.R. Wang, J.T. Lizier, and M. Prokopenko. Fisher information at the edge of chaos in random Boolean networks. *Artificial Life*, 17(4):315–329, 2011.

[16] M. Prokopenko, J.T. Lizier, O. Obst, and X.R. Wang. Relating Fisher information to order parameters. *Physical Review E*, 84:041116, 2011.

[17] M. Prokopenko and I. Einav. Information thermodynamics of near-equilibrium computation. *Physical Review E*, 91:062143, 2015.